A. Unproven theorem.

<u>Thm</u>. G top. group.

There is a top. space BG and a "universal" prin. G-bundle $EG \to BG$ s.t. for any paracompact Hausdorff space

$$\boxed{[X, BG]} \xrightarrow{\simeq} \boxed{\mathrm{Tors}_G(X)}.$$

$$[f] \longmapsto f^*(EG \to BG).$$

includes CW cplxes, manifolds, etc.

<u>Rem</u>. $\Omega BG \simeq G$.

<u>Exs</u> (a) G is <u>discrete</u>

$\Rightarrow$ BG classifies G-Galois covers and

$$BG \simeq K(G,1).$$

(b) A discrete abelian group, $K(A, n-1)$ top. ab. gp. and

$$BK(A, n-1) \simeq K(A, n).$$

(c) $\mathbb{Z}$-torsors $\rightsquigarrow$ $B\mathbb{Z} \simeq S^1 \simeq K(\mathbb{Z}, 1)$.

S^1-torsors $\rightsquigarrow$ $BS^1 \simeq \mathbb{CP}^\infty \simeq K(\mathbb{Z}, 2)$

(d) $\mathbb{Z}/2$ torsors $\rightsquigarrow$ $B\mathbb{Z}/2 \simeq \mathbb{RP}^\infty$.

<u>Cor</u>. • $\mathbb{Z}$-torsors on $X \leftarrow [X, B\mathbb{Z}] \simeq [X, S^1] \simeq [X, K(\mathbb{Z}, 1)] \simeq H^1(X, \mathbb{Z})$.

• S^1-torsors

$$[X, BS^1] \simeq [X, \mathbb{CP}^\infty] \simeq [X, K(\mathbb{Z}, 2)] \simeq H^2(X, \mathbb{Z})$$

<u>Rem</u>. $[X, K(A, n)] \simeq [X, \Omega K(A, n+1)]$

$\simeq [X, \Omega^2 K(A, n+2)]$

B. Some groups.

$\mathbb{C}^{n^2}$ $\supseteq$	maximal compact	split real forms
$GL_n(\mathbb{C})$	U_n $(\bar{X}^t X = I_n)$	$Sp_{2n}(\mathbb{R})$
$GL_1(\mathbb{C}) = \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$	S^1	
$SL_n(\mathbb{C})$	SU_n	
$O_n(\mathbb{C})$ auts of a rank n non-degen symmetric bilin form	O_n $X^t X = I_n$	$GL_n(\mathbb{R})$
$SO_n(\mathbb{C})$	SO_n	$SL_n(\mathbb{R})$
$Sp_{2n}(\mathbb{C})$ auts of a non-degenerate skew-symmetric bilin form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\oplus n}$.	$Sp(2n)$ preserve a quaternionic hermitian form	

Thm. In any row, the groups are homotopy eq. and have homotopy eq. classifying spaces.

Cor. X paracompact Hausdorff. $H \hookrightarrow G$

(a)
$$\begin{tikzcd}
{[X, BU_n]} \arrow[r, "\simeq"] \arrow[d, phantom, "\cong" rotate=90] & {[X, BGL_n(\mathbb{C})]} \arrow[l, bend right, dashed, "\text{red. of str. gp.}"'] \arrow[d, phantom, "\cong" rotate=90] \\
\text{rank } n \text{ top. } \mathbb{C}\text{-v.b. on } X \text{ w/ a non-degenerate hermitian form} & \text{rank } n \text{ top. } \mathbb{C}\text{-v.b. on } X
\end{tikzcd}$$

(b) $[X, BO_n(\mathbb{C})] \xleftarrow{\simeq} [X, BO_n] \xrightarrow{\simeq} [X, BGL_n(\mathbb{R})]$

+ non-deg. symm bilinear form (aka metric)

rank n $\mathbb{R}$-v.bs.

C. A warning.

Notation. G top. group,

$G^\delta = G$ as a set w/ discrete top.

$$G^\delta \longrightarrow G$$

continuous homomorphism

Warning. G-torsors are <u>not</u> the same as G^δ-torsors.

Ex. $X = U \cup V$, $W = U \cap V$.

$E \xrightarrow{G\text{-tors}} X$ triv. on U, V,

$$W \xrightarrow{\text{cont}} G^\delta.$$

loc. constant function.

Def. Principal G^δ-bundles are called locally constant principal G-bundles.

$$X \longrightarrow \tau_{\le 1} X \quad \text{fundamental groupoid}$$

Thm.
$$\mathrm{Tors}^{lc}_{G}(X) \cong \mathrm{Tors}_{G^\delta}(X) \cong \mathrm{Tors}_{G^\delta}(\tau_{\le 1} X)$$
$$\cong \;(X \text{ connected})\; \mathrm{Hom}_{\mathrm{Groups}}(\pi_1 X, G)/G.$$

Proof. G^δ is discrete, so BG^δ is a $K(G^\delta, 1)$.

So, BG^δ is in $\mathcal{S}_{\le 1}$.

$$\mathcal{S} \underset{\tau_{\le 1}}{\rightleftarrows} \mathcal{S}_{\le 1}$$

$$[X, BG^\delta] \simeq [\tau_{\le 1} X, BG^\delta]$$

$$\mathrm{Hom}_{\mathcal{S}}(X, BG^\delta) \simeq \mathrm{Hom}_{\mathcal{S}_{\le 1}}(\tau_{\le 1} X, BG^\delta)$$

$$= \mathrm{Hom}_{\mathrm{Groupoids}}(\pi_1 X, G).$$

$$E|_x \cong G^\delta$$

Rem. Princ. G^{δ}-bundles trivialized at $x \in X$,

$\mathrm{Hom}_{\mathrm{Groups}}(\pi_1 X, G)$.

Ex. Locally constant princ. $GL_n(\mathbb{C})$-bundles trivialized at $x \in X$ $\longleftrightarrow$ representations of $\pi_1(X, x)$ in $GL_n(\mathbb{C})$.

(Other terminology: local systems, or flat v.b.)

Q. How do we know that there are non-flat vbs?

Ex. S^2 T_{S^2} non-trivial ($\not\cong \mathbb{R}^2 \times S^2$)

$\pi_1 S^2 = 0$

$\Rightarrow T_{S^2}$ is not lc.

Ex. $\mathbb{P}^1_{\mathbb{C}}$ $\mathcal{O}(n)$ are non-trivial for $n \neq 0$.

princ. $\mathbb{C}^\times$-bundles w/o red. of str. grp to $\mathbb{C}^{\times,\delta}$.

$(\pi_1 \mathbb{P}^1_{\mathbb{C}} = 0)$.

Conjecture (Friedlander-Milnor).

$$H^*(BG(\mathbb{C}), \mathbb{Z}/\ell) \xrightarrow{\simeq} H^*(BG(\mathbb{C})^\delta, \mathbb{Z}/\ell)$$

$\forall$ primes ℓ.

(complex algebraic G.)

Ex. $BGL_n(\mathbb{C}) \simeq Gr_n(\mathbb{C}^\infty)$.

Thm. $X/\mathbb{C}$ sm?

$$H^*_{et}(X, \mathbb{Z}/\ell) \simeq H^*(X(\mathbb{C}), \mathbb{Z}/\ell)$$

Artin

$$W(\mathbb{F}_p[x]) \hookrightarrow W(\mathbb{F}_p[x^{1/p^\infty}]) \cong \mathbb{Z}_p[x^{1/p^\infty}]_p^\wedge$$

$$-\left([x],\ V[x],\ V^2[x],\ \dots\ \right).$$